

REAL ZEROS OF HURWITZ-LERCH ZETA AND HURWITZ-LERCH TYPE OF EULER-ZAGIER DOUBLE ZETA FUNCTIONS

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ABSTRACT. Let $0 < a \leq 1$, $s, z \in \mathbb{C}$ and $0 < |z| \leq 1$. Then the Hurwitz-Lerch zeta function is defined by $\Phi(s, a, z) := \sum_{n=0}^{\infty} z^n (n+a)^{-s}$ when $\sigma := \Re(s) > 1$. In this paper, we show that the Hurwitz zeta function $\zeta(\sigma, a) := \Phi(\sigma, a, 1)$ does not vanish for all $0 < \sigma < 1$ if and only if $a \geq 1/2$. Moreover, we prove that $\Phi(\sigma, a, z) \neq 0$ for all $0 < \sigma < 1$ and $0 < a \leq 1$ when $z \neq 1$. Real zeros of Hurwitz-Lerch type of Euler-Zagier double zeta functions are studied as well.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Reals zeros of Hurwitz-Lerch zeta functions. As one of a generalization of the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, the following function is well-known.

Definition 1.1 (see [8, p. 53, (1)]). *For $0 < a \leq 1$, $s, z \in \mathbb{C}$ and $0 < |z| \leq 1$, the Hurwitz-Lerch zeta function $\Phi(s, a, z)$ is defined by*

$$\Phi(s, a, z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad s := \sigma + it, \quad \sigma > 1, \quad t \in \mathbb{R}. \quad (1.1)$$

Note that the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$ are written by $\Phi(s, 1, 1)$ and $\Phi(s, a, 1)$, respectively. The Dirichlet series of $\Phi(s, a, z)$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. The function $\Phi(s, a, z)$ with $z \neq 1$ is analytically continuable to the whole complex plane but $\zeta(s, a)$ is a meromorphic function with a simple pole at $s = 1$. In the present paper, we show the following theorem.

Theorem 1.2. *We have the following:*

- (1). *Let $z = 1$. Then $\Phi(\sigma, a, 1) \neq 0$ for all $0 < \sigma < 1$ if and only if $a \geq 1/2$.*
- (2). *Let $z \neq 1$. Then $\Phi(\sigma, a, z) \neq 0$ for all $0 < \sigma < 1$ and $0 < a \leq 1$.*

When $z = 1$, the Hurwitz zeta function $\zeta(\sigma, a) := \Phi(\sigma, a, 1) > 0$ for $\sigma > 1$ from the series expression $\sum_{n=0}^{\infty} (n+a)^{-\sigma}$. Berndt showed that $\zeta(s, a) - a^{-s}$ has no zeros on $|s-1| \leq 1$ when $0 \leq a \leq 1$ in [6, Theorem 3]. Note that $\zeta(\sigma, a) - a^{-\sigma} = \zeta(s)$ if $a = 0$ and [6, Theorem 3] mentioned above implies that

$$\zeta(\sigma, a+1) = \zeta(\sigma, a) - a^{-\sigma} \neq 0$$

for any $0 < \sigma < 1$ and $0 \leq a \leq 1$. In [20, Theorem 3], Spira proved that if $\sigma \leq -4a - 1 - 2[1 - 2a]$ and $|t| \leq 1$, then $\zeta(s, a) \neq 0$ except for zeros on the negative real line, one in each interval $(-2n - 4a - 1, -2n - 4a + 1)$, where $n \in \mathbb{N}$ and $n \geq 1 - 2a$. Some

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analogous results for the Lerch zeta function $\Phi(s, a, e^{2\pi i\theta})$, where $0 < \theta \leq 1$ are proved by Garunkštis and Laurinćikas in [9] (see also [12, Section 8]). Recently, Schipani [18] showed that $\zeta(\sigma, a)$ has no zeros and is actually negative for $0 < \sigma < 1$ and $1 - \sigma \leq a$. Note that we prove that $\zeta(\sigma, a) < 0$ for any $0 < \sigma < 1$ when $a \geq 1/2$ during the proof process of Theorem 1.4 (see (2.8)). Denote the polylogarithm by $\text{Li}_s(z) := z\Phi(s, 1, z) = \sum_{n=1}^{\infty} z^n n^{-s}$. More than 100 years ago, Roy [17] proved that $\text{Li}_\sigma(z) \neq 0$ for all $|z| \leq 1$, $z \neq 1$ and $\sigma > 0$ (for zeros of polylogarithms $\text{Li}_s(z)$, we can refer to [16, Section 8]).

Let $L(s, \chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$ be the Dirichlet L -function with a Dirichlet character χ . About 80 years ago, Siegel [19] (see also [15, Theorem 11.11]) showed for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that, if χ is a real primitive Dirichlet character modulo q , then $L(1, \chi) > C_\varepsilon q^{-\varepsilon}$. It is expected that $L(\sigma, \chi) \neq 0$ for all $0 < \sigma < 1$. Namely, it is conjectured that so-called Siegel zeros of Dirichlet L -functions do not exist. Let φ be the Euler totient function and χ be a primitive Dirichlet character of conductor of q . Then the following relations between Dirichlet L -functions and Hurwitz zeta functions are well-known.

$$L(s, \chi) = \sum_{r=1}^q \sum_{n=0}^{\infty} \frac{\chi(r+nq)}{(r+nq)^s} = \sum_{r=1}^q \chi(r) \sum_{n=0}^{\infty} \frac{1}{(r+nq)^s} = q^{-s} \sum_{r=1}^q \chi(r) \zeta(s, r/q),$$

$$\zeta(s, r/q) = \sum_{n=0}^{\infty} \frac{1}{(n+r/q)^s} = \sum_{n=0}^{\infty} \frac{q^s}{(r+qn)^s} = \frac{q^s}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(r) L(s, \chi).$$

The relations between Hurwitz zeta functions and polylogarithms are expressed as

$$\zeta(s, r/q) = \sum_{n=1}^q e^{-2\pi i r n/q} \text{Li}_s(e^{2\pi i r n/q}), \quad \text{Li}_s(e^{2\pi i r/q}) = q^{-s} \sum_{n=1}^q e^{2\pi i r n/q} \zeta(s, n/q).$$

Hence, we have the following relations between Dirichlet L -functions and polylogarithms

$$L(s, \chi) = \frac{1}{G(\bar{\chi})} \sum_{n=1}^q \sum_{r=1}^q \frac{\bar{\chi}(r) e^{2\pi i r n/q}}{n^s} = \frac{1}{G(\bar{\chi})} \sum_{r=1}^q \bar{\chi}(r) \text{Li}_s(e^{2\pi i r/q}),$$

$$\text{Li}_s(e^{2\pi i r/q}) = q^{-s} \sum_{n=1}^q e^{2\pi i r n/q} \zeta(s, n/q) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} G(\bar{\chi}) L(s, \chi),$$

where $G(\bar{\chi}) := \sum_{n=1}^q \bar{\chi}(n) e^{2\pi i r n/q}$ denotes the Gauss sum associated to $\bar{\chi}$. It should be emphasized that we have $\zeta(\sigma, a) \neq 0$ for all $0 < \sigma < 1$ if and only if $a \geq 1/2$ and $\text{Li}_\sigma(z) \neq 0$ for all $0 < \sigma < 1$ and $|z| \leq 1$ from Theorem 1.2 despite of the six relations above and the difficulty of the Siegel zero's problem (see also the paper [7] by Conrey, Granville, Poonen and Soundararajan).

1.2. Reals zeros of Hurwitz-Lerch type of Euler-Zagier double zeta functions.

As a double sum and two variable version of the Hurwitz-Lerch zeta function $\Phi(s, a, z)$, we define the following function.

Definition 1.3 (see [11, (1)]). For $0 < a \leq 1$, $s_1, s_2, z_1, z_2 \in \mathbb{C}$ and $0 < |z_1|, |z_2| \leq 1$, the Hurwitz-Lerch type of Euler-Zagier double zeta function $\Phi_2(s_1, s_2, a, z_1, z_2)$ is defined by

$$\Phi_2(s_1, s_2, a, z_1, z_2) := \sum_{m=0}^{\infty} \frac{z_1^m}{(m+a)^{s_1}} \sum_{n=1}^{\infty} \frac{z_2^{n-1}}{(m+n+a)^{s_2}}. \quad (1.2)$$

The function $\Phi_2(s_1, s_2, a, z_1, z_2)$ can be continued meromorphically to the whole space \mathbb{C}^2 by Komori's result [11, Theorem 3.14] (see also Lemma 2.8 and Proposition 2.6). In this paper, we prove the following theorem.

Theorem 1.4. *We have the following:*

- (1). *Let $z_1 = z_2 = 1$. Then $\Phi_2(\sigma_1, \sigma_2, a, 1, 1) \neq 0$ for all $0 < \sigma_1 < 1$, $\sigma_2 > 1$ and $1 < \sigma_1 + \sigma_2 < 2$ if and only if $a \geq 1/2$.*
- (2). *Let $z_1 = 1$ and $z_2 \neq 1$. Then $\Phi_2(\sigma_1, \sigma_2, a, 1, z_2) \neq 0$ for all $\sigma_1 > 1$, $\sigma_2 > 0$ and $0 < a \leq 1$.*
- (3). *Let $z_1 \neq 1$ and $z_2 = 1$. Then $\Phi_2(\sigma_1, \sigma_2, a, z_1, 1) \neq 0$ for all $\sigma_1 > 0$, $\sigma_2 > 1$ and $0 < a \leq 1$.*
- (4). *Let $z_1 \neq 1$ and $z_2 \neq 1$. Then $\Phi_2(\sigma_1, \sigma_2, a, z_1, z_2) \neq 0$ for all $\sigma_1 > 0$, $\sigma_2 > 0$ and $0 < a \leq 1$.*

When $z_1 = z_2 = 1$, the function above can be regarded as a special case of [13, (3.2)] which gives an application to special values of Hecke L -series of real quadratic fields. Note that Atkinson [5] obtained an analytic continuation for $\zeta_2(s_1, s_2; a) := \Phi_2(s_1, s_2, a, z_1, z_2)$ with $a = 1$ in order to study the mean square $\int_0^T |\zeta(1/2 + it)|^2 dt$ more than 60 years ago. Matsumoto [14] gave not only an analytic continuation to whole \mathbb{C}^2 plane but also a functional equation for this kind of zeta functions. Zeros of the Hurwitz type of Euler-Zagier double zeta function $\zeta_2(s_1, s_2; a)$ (after the continuation) at negative integer points are discussed by Akiyama, Egami and Tanigawa [1], Akiyama and Tanigawa [3], Kelliher and Masri [10] and Zhao [21]. We have to remark that Theorem 1.4 (1) indicates the existence of a real zero of $\zeta_2(s_1, s_2; a)$ off negative integer points when $0 < a < 1/2$. Related to this problem, we have the following proposition.

Proposition 1.5. *The function $\zeta_2(\sigma, \sigma; a)$ has at least one real zero for $1/2 < \sigma < 1$. Hence, there exist $1/2 < \sigma_1, \sigma_2 < 1$ such that $\zeta_2(\sigma_1, \sigma_2; a) = 0$.*

2. PROOFS

2.1. Proof of Theorem 1.2. In order to prove (1) of Theorem 1.2, we define $H(a, x)$ by

$$H(a, x) := \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} = \frac{xe^{(1-a)x} - e^x + 1}{x(e^x - 1)}, \quad x > 0. \quad (2.1)$$

Lemma 2.1. *For $0 < \sigma < 1$ we have the integral representation*

$$\Gamma(s)\zeta(s, a) = \int_0^\infty \left(\frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx = \int_0^\infty H(a, x) x^{s-1} dx. \quad (2.2)$$

Proof. When $\sigma > 1$, it is well-known that

$$\Gamma(s)\zeta(s, a) = \int_0^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - 1} dx$$

(see [4, Theorem 12.2]). Hence we have

$$\begin{aligned}
\Gamma(s)\zeta(s, a) &= \int_0^1 \frac{x^{s-1}e^{(1-a)x}}{e^x - 1} dx + \int_1^\infty \frac{x^{s-1}e^{(1-a)x}}{e^x - 1} dx \\
&= \int_0^1 \left(\frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx + \int_0^1 x^{s-2} dx + \int_1^\infty \frac{x^{s-1}e^{(1-a)x}}{e^x - 1} dx \\
&= \int_0^1 \left(\frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx + \int_1^\infty \frac{x^{s-1}e^{(1-a)x}}{e^x - 1} dx + \frac{1}{s-1}.
\end{aligned} \tag{2.3}$$

By the Taylor expansion of e^x , we have

$$\begin{aligned}
H(a, x) &= \frac{x(\sum_{n=0}^\infty (1-a)^n x^n / n!) - \sum_{n=1}^\infty x^n / n!}{x(\sum_{n=1}^\infty x^n / n!)} \\
&= \frac{(1/2 - a)x^2 + ((1-a)^2/2! - 1/3!)x^3 + \dots}{x^2 + x^3/2! + \dots}.
\end{aligned} \tag{2.4}$$

Hence, for $\sigma > 0$, it holds that

$$\int_0^1 \left| \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right| |x^{s-1}| dx \ll \int_0^1 x^{\sigma-1} dx < \infty. \tag{2.5}$$

On the other hand, we have

$$\frac{1}{s-1} = - \int_1^\infty \frac{x^{s-1}}{x} dx, \quad 0 < \sigma < 1.$$

Moreover, for $0 < \sigma < 1$ one has

$$\begin{aligned}
\int_1^\infty \left| \frac{e^{(1-a)x}}{e^x - 1} \right| |x^{s-1}| dx &\ll \int_1^\infty \frac{e^{(1-a)x}}{e^x - 1} x^{\sigma-1} dx < \infty, \\
\int_1^\infty |x^{s-2}| dx &= \int_1^\infty x^{\sigma-2} dx = \frac{1}{1-\sigma} < \infty.
\end{aligned} \tag{2.6}$$

Therefore, the integral representation

$$\Gamma(s)\zeta(s, a) = \int_0^1 \left(\frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx + \int_1^\infty \frac{x^{s-1}e^{(1-a)x}}{e^x - 1} dx - \int_1^\infty \frac{x^{s-1}}{x} dx$$

gives an analytic continuation for $0 < \sigma < 1$. Thus we obtain this Lemma. \square

Lemma 2.2. *The function $H(a, x)$ defined by (2.1) is negative for all $x > 0$ if and only if $a \geq 1/2$.*

Proof. First suppose $0 < a < 1/2$. Then we have $\lim_{x \rightarrow +0} H(a, x) = 1/2 - a > 0$ by (2.4). Besides, one has

$$h(a, x) := x(e^x - 1)H(a, x) = xe^{(1-a)x} - e^x + 1 < 0$$

when x is sufficiently large by $e^{1-a} < e$. Hence the function $H(a, x)$ is not negative definite when $0 < a < 1/2$.

Next suppose $a \geq 1/2$. Obviously, we have $x(e^x - 1) > 0$ for all $x > 0$. Thus we only have to consider $h(a, x)$ which is the numerator of $H(a, x)$. It holds that $h(a, 0) = 0$. Thus we show the inequality

$$h'(a, x) = (1-a)xe^{(1-a)x} + e^{(1-a)x} - e^x < 0, \quad x > 0.$$

This inequality is equivalent to $(1-a)xe^{(1-a)x} + e^{(1-a)x} < e^x$, namely, $1 + (1-a)x < e^{ax}$. We can prove this inequality by the assumption $1-a \leq a$ and the Taylor expansion of $e^{ax} = \sum_{n=0}^{\infty} (ax)^n/n!$. \square

Proof of (1) of Theorem 1.2. Let $0 < a < 1/2$. Then we have

$$\zeta(0, a) = \frac{1}{2} - a > 0$$

(see [4, p. 268]). Moreover, for any integer $N \geq 0$ and $\sigma > 0$, we have

$$\zeta(s, a) = \sum_{n=0}^N \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s \int_N^{\infty} \frac{x-[x]}{(x+a)^{s+1}} dx, \quad (2.7)$$

where $[x]$ denotes the maximal integer less than or equal to x (see [4, Theorem 12.21]). Thus it holds that $\zeta(\sigma, a) \in \mathbb{R}$ when $\sigma \in (0, 1)$ and

$$\lim_{\sigma \rightarrow 1-0} \zeta(\sigma, a) = -\infty.$$

Hence $\zeta(s, a)$ has at least one zero in the interval $(0, 1)$ when $0 < a < 1/2$.

Secondly suppose $a \geq 1/2$. Then we have

$$\Gamma(\sigma)\zeta(\sigma, a) = \int_0^{\infty} \left(\frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{\sigma-1} dx, \quad 0 < \sigma < 1$$

by the integral representation (2.2). It is well-known that $\Gamma(\sigma) > 0$ for any $0 < \sigma < 1$. Thus we obtain

$$\zeta(\sigma, a) < 0 \quad (2.8)$$

for all $0 < \sigma < 1$ by Lemma 2.2 and the integral representation above. Therefore $\zeta(\sigma, a)$ does not vanish in the interval $(0, 1)$ when $a \geq 1/2$. \square

Next we quote the following integral representation of Hurwitz-Lerch zeta function $\Phi(s, a, z)$ to show (2) of Theorem 1.2.

Lemma 2.3 (see [8, p. 53, (3)]). *When $z \neq 1$, we have*

$$\Phi(s, a, z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx, \quad \Re(s) > 0. \quad (2.9)$$

Proof of (2) of Theorem 1.2. It should be noted that the integral representation (2.9) converges absolutely for $\sigma > 0$ when $z \neq 1$ since one has $e^x - z \neq 0$ for any $x \geq 0$ and

$$|\Phi(s, a, z)\Gamma(s)| \leq \int_0^1 \frac{x^{\sigma-1} e^{(1-a)x}}{|e^x - z|} dx + \int_1^{\infty} \frac{x^{\sigma-1} e^{(1-a)x}}{|e^x - z|} dx < \infty. \quad (2.10)$$

First suppose $z \in [-1, 1)$. Then we have $e^{(1-a)x} > 0$ and $e^x - z > 0$ for all $x \geq 0$. Hence, for any $\sigma > 0$, $0 < a \leq 1$ and $z \in [-1, 1)$, we have

$$\Phi(\sigma, a, z) > 0. \quad (2.11)$$

Next suppose z is not real. Then it holds that

$$\Phi(s, a, z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{(1-a)x} (e^x - \bar{z})}{|e^x - z|^2} dx, \quad \Re(s) > 0,$$

where \bar{z} is the complex conjugate of z , from (2.9). Obviously we have

$$\Im(e^{(1-a)x}(e^x - \bar{z})) = -e^{(1-a)x}\Im(\bar{z}) \begin{cases} > 0 & \text{if } \Im(\bar{z}) < 0 \\ < 0 & \text{if } \Im(\bar{z}) > 0 \end{cases} \quad (2.12)$$

for all $x > 0$. Therefore, it holds that

$$\Im(\Phi(\sigma, a, z)) \neq 0$$

for any $\sigma > 0$ and $0 < a \leq 1$ when z is not real. \square

2.2. Proofs of Theorem 1.4 and Proposition 1.5. First, we show that the series expression (1.2) with $z_1 = z_2 = 1$ converges absolutely when $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$.

Lemma 2.4. *For $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$, the series*

$$\sum_{m=0}^{\infty} \frac{1}{(m+a)^{s_1}} \sum_{n=1}^{\infty} \frac{1}{(m+n+a)^{s_2}} = \sum_{0 \leq n_1 < n_2} \frac{1}{(n_1+a)^{s_1}(n_2+a)^{s_2}} \quad (2.13)$$

converges absolutely.

Proof. For any $\varepsilon > 0$, it holds that

$$\sum_{n_2 > n_1 \geq 0} \frac{1}{(n_2+a)^{s_2}(n_1+a)^{s_1}} = \sum_{n_2=1}^{\infty} \frac{\sum_{n_1=0}^{n_2-1} (n_1+a)^{-s_1}}{(n_2+a)^{s_2}} \ll \sum_{n_2=1}^{\infty} \frac{\max\{1, n_2^{1-s_1+\varepsilon}\}}{(n_2+a)^{s_2}}.$$

Hence the series above converges absolutely in the region $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$. \square

We prove the following integral representation of $\Gamma(s_1)\Gamma(s_2)\zeta(s_1, s_2; a)$ with $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$ which is a special case of [13, (3.4)].

Lemma 2.5. *For $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$, we have the integral representation*

$$\Gamma(s_1)\Gamma(s_2)\zeta_2(s_1, s_2; a) = \int_0^\infty \frac{y^{s_2-1}}{e^y - 1} \int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - 1} dx dy. \quad (2.14)$$

Proof. For reader's convenience, we prove Lemma 2.5 although we obtain a proof of this lemma by the proof in [13, p. 391]. When $y > 0$ and $\sigma_1 > 0$, one has

$$\int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - 1} dx = \sum_{m=0}^{\infty} \int_0^\infty e^{-(a+m)(x+y)} x^{s_1-1} dx = \Gamma(s_1) \sum_{m=0}^{\infty} e^{-(a+m)y} (a+m)^{-s_1}.$$

Hence, if $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$, we have

$$\int_0^\infty \frac{y^{s_2-1}}{e^y - 1} \int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - 1} dx dy = \Gamma(s_1) \sum_{m=0}^{\infty} (a+m)^{-s_1} \int_0^\infty \frac{y^{s_2-1} e^{-(a+m)y}}{e^y - 1} dy$$

where the change of the integration and the summation is justified by Lebesgue's dominated convergence theorem and

$$\begin{aligned} & \sum_{m=0}^{\infty} (a+m)^{-\sigma_1} \int_0^{\infty} \frac{y^{\sigma_2-1} e^{-(a+m)y}}{|e^y - 1|} dy \\ & \ll \sum_{m=0}^{\infty} (a+m)^{-\sigma_1} \int_0^1 e^{-(a+m+1)y} y^{\sigma_2-2} dy + \sum_{m=0}^{\infty} (a+m)^{-\sigma_1} \int_1^{\infty} e^{-(a+m+1)y} y^{\sigma_2-1} dy \\ & \ll \Gamma(\sigma_2 - 1) \sum_{m=0}^{\infty} (a+m)^{-\sigma_1-\sigma_2+1} + \Gamma(\sigma_2) \sum_{m=0}^{\infty} (a+m)^{-\sigma_1-\sigma_2}. \end{aligned}$$

Moreover, we have

$$\int_0^{\infty} \frac{y^{s_2-1} e^{-(a+m)y}}{e^y - 1} dy = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-(a+m+n)y} y^{s_2-1} dy = \Gamma(s_2) \sum_{n=1}^{\infty} (a+m+n)^{-s_2}$$

when $\Re(s_2) > 1$. Therefore we obtain this lemma. \square

When $0 < \Re(s_1) < 1$, $\Re(s_2) > 1$ and $1 < \Re(s_1 + s_2) < 2$, we have the following integral representation of $\Gamma(s_1)\Gamma(s_2)\zeta_2(s_1, s_2; a)$ which is a key for the proof of (1) of Theorem 1.4. It should be mentioned that the series (2.13) with $z_1 = z_2 = 1$ does not converge absolutely in this case since we have

$$\sum_{n_2 > n_1 \geq 0} \frac{1}{(n_2 + a)^{\sigma_2} (n_1 + a)^{\sigma_1}} = \sum_{n_2=1}^{\infty} \frac{\sum_{n_1=0}^{n_2-1} (n_1 + a)^{-\sigma_1}}{(n_2 + a)^{\sigma_2}} \gg \sum_{n_2=1}^{\infty} \frac{n_2^{1-\sigma_1}}{(n_2 + a)^{\sigma_2}} = \infty.$$

Proposition 2.6. *For $0 < \Re(s_1) < 1$, $\Re(s_2) > 1$ and $1 < \Re(s_1 + s_2) < 2$, we have the integral representation*

$$\Gamma(s_1)\Gamma(s_2)\zeta_2(s_1, s_2; a) = \int_0^{\infty} \frac{y^{s_2-1}}{e^y - 1} \int_0^{\infty} H(a, x+y) x^{s_1-1} dx dy + \Gamma(s_1)\Gamma(1-s_1) \int_0^{\infty} H(1, y) y^{s_1+s_2-2} dy, \quad (2.15)$$

where $H(a, x)$ is defined by (2.1). Moreover, one has the integral representation

$$\Gamma(s_1)\Gamma(s_2)\zeta_2(s_1, s_2; a) = \int_0^{\infty} \int_0^{\infty} \frac{y^{s_2-1}}{e^y - 1} H(a, x+y) x^{s_1-1} dx dy + \int_0^{\infty} \int_0^{\infty} \frac{x^{s_1-1}}{x+y} H(1, y) y^{s_2-1} dx dy \quad (2.16)$$

for $0 < \Re(s_1) < 1$, $\Re(s_2) > 1$ and $1 < \Re(s_1 + s_2) < 2$.

Proof. It is known (see for example [13, p. 392]) that

$$\int_0^{\infty} \frac{x^{s_1-1}}{x+y} dx = y^{s_1-1} \Gamma(s_1) \Gamma(1-s_1), \quad y > 0, \quad 0 < \Re(s_1) < 1. \quad (2.17)$$

Hence, for $0 < \Re(s_1) < 1$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$, we have

$$\begin{aligned} & \Gamma(s_1)\Gamma(s_2)\zeta(s_1, s_2; a) \\ & = \int_0^{\infty} \frac{y^{s_2-1}}{e^y - 1} \int_0^{\infty} \left(\frac{e^{(1-a)(x+y)}}{e^{x+y} - 1} - \frac{1}{x+y} \right) x^{s_1-1} dx dy + \int_0^{\infty} \frac{y^{s_2-1}}{e^y - 1} \int_0^{\infty} \frac{x^{s_1-1}}{x+y} dx dy \\ & = \int_0^{\infty} \frac{y^{s_2-1}}{e^y - 1} \int_0^{\infty} H(a, x+y) x^{s_1-1} dx dy + \Gamma(s_1)\Gamma(1-s_1) \int_0^{\infty} \frac{y^{s_1+s_2-2}}{e^y - 1} dy \end{aligned}$$

by Lemma 2.5. It should be noted that this formula is a special case of [13, (3.8)] in the region $0 < \Re(s_1) < 1$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$. Now we show the first integral in the formula above converges absolutely when $0 < \Re(s_1) < 1$ and $\Re(s_2) > 1$. This is proved as follows. Divide it into two integrals $\int_0^1 \int_0^1$ and $\iint_{\mathbb{R}_+^2 \setminus D_1}$, where \mathbb{R}_+ is the set of all positive real numbers and $D_1 := \{x, y \in \mathbb{R} : 0 < x, y \leq 1\}$. Then we have

$$\int_0^1 \int_0^1 \frac{y^{\sigma_2-1}}{e^y - 1} |H(a, x+y)| x^{\sigma_1-1} dx dy < \infty$$

by using (2.4). Obviously, we have $|H(a, x+y)| \ll (x+y)^{-1} < x^{-1}$ when $x \geq 1$ and $y > 0$, and $|H(a, x+y)| \ll (x+y)^{-1} < y^{-1}$ when $0 < x < 1$ and $y \geq 1$. Hence one has

$$\begin{aligned} \iint_{\mathbb{R}_+^2 \setminus D_1} \frac{y^{\sigma_2-1}}{e^y - 1} |H(a, x+y)| x^{\sigma_1-1} dx dy &\ll \int_1^\infty \int_1^\infty \frac{y^{\sigma_2-1}}{e^y - 1} x^{\sigma_1-2} dx dy \\ &+ \int_1^\infty \frac{y^{\sigma_2-2}}{e^y - 1} dy \int_0^1 x^{\sigma_1-1} dx dy + \int_0^1 \frac{y^{\sigma_2-2}}{e^y - 1} dy \int_1^\infty x^{\sigma_1-2} dx < \infty. \end{aligned}$$

Next consider the second integral of (2.15). From the view of (2.3), one has

$$\int_0^\infty \frac{y^{s_1+s_2-2}}{e^y - 1} dy = \int_0^1 H(1, y) y^{s_1+s_2-2} dy + \int_1^\infty \frac{y^{s_1+s_2-2}}{e^y - 1} dy + \frac{1}{s_1 + s_2 - 2}.$$

The two integrals in the formula above converges absolutely when $1 < \Re(s_1 + s_2) < 2$ by (2.5) and (2.6). On the other hand, we have

$$\frac{1}{s_1 + s_2 - 2} = - \int_1^\infty \frac{y^{s_1+s_2-2}}{y} dy, \quad 1 < \Re(s_1 + s_2) < 2.$$

Obviously, the integral $\int_1^\infty y^{s_1+s_2-3} dy$ converges absolutely when $1 < \Re(s_1 + s_2) < 2$. Therefore, we obtain (2.15) by the definition of $H(1, y)$.

It was shown in the proof of (2.15) that the first double integrals converges absolutely when $0 < \Re(s_1) < 1$ and $\Re(s_2) > 1$. If we can interchange of the order of the second double integrations, we have (2.16) from (2.15) and (2.17). This is justified as follows. By using (2.15), we have

$$\int_0^\infty \left| \frac{x^{\sigma_1-1}}{x+y} \right| dx = \int_0^\infty \frac{x^{\sigma_1-1}}{x+y} dx = y^{\sigma_1-1} \Gamma(\sigma_1) \Gamma(1-\sigma_1) < \infty$$

for $y > 0$ and $0 < \sigma_1 < 1$. Furthermore, it holds that

$$\int_0^\infty \left| \frac{H(1, y)}{x+y} y^{s_1+s_2-1} \right| dy \leq \int_0^\infty H(1, y) y^{\sigma_1+\sigma_2-2} dy < \infty$$

when $x \geq 0$ and $1 < \Re(s_1 + s_2) < 2$ from the view of the proof of (2.5) and (2.6). Thus we can apply Fubini's theorem. \square

We quote the following Lemma from Akiyama and Ishikawa [2]. We have to remark that Akiyama and Ishikawa [2] consider not $\sum_{0 \leq n_1 < n_2}$ but $\sum_{0 < n_1 < n_2}$ in the definition of the double zeta-function. Note that we have

$$\begin{aligned} \zeta_2(s_1, s_2; a) &= \sum_{0 < n_2} \frac{1}{a^{s_1}(n_2 + a)^{s_2}} + \sum_{0 < n_1 < n_2} \frac{1}{(n_1 + a)^{s_1}(n_2 + a)^{s_2}} \\ &= a^{-s_1} (\zeta(s_2, a) - a^{-s_2}) + \sum_{0 < n_1 < n_2} \frac{1}{(n_1 + a)^{s_1}(n_2 + a)^{s_2}}. \end{aligned}$$

Lemma 2.7 (see [2, (15)]). *Let $\lambda > 0$, $l \in \mathbb{N}_0$, $\tilde{B}_l(x) := B_l(x - [x])$, where $B_l(x)$ be the l -th Bernoulli polynomial and*

$$\Phi_l(s | \lambda, a) := \frac{(s)_{l+1}}{(l+1)!} \int_{\lambda}^{\infty} \frac{\tilde{B}_{l+1}(x)}{(x+a)^{s+l+1}} dx, \quad (s)_l := \begin{cases} s(s+1) \cdots (s+l-1) & l \geq 1, \\ 1 & l = 0. \end{cases}$$

Then, for $\Re(s_1 + s_2) > -l$, we have

$$\begin{aligned} \zeta_2(s_1, s_2; a) &= a^{-s_1} (\zeta(s_2, a) - a^{-s_2}) + \frac{\zeta(s_1 + s_2 - 1, a) - a^{1-s_1-s_2}}{s_2 - 1} \\ &\quad + \sum_{r=0}^l \frac{B_{r+1}(0)}{(r+1)!} (s_2)_r (\zeta(s_1 + s_2 + r, a) - a^{-s_1-s_2-r}) - \sum_{n=1}^{\infty} \frac{\Phi_l(s_2 | n, a)}{(n+a)^{s_1}}. \end{aligned} \quad (2.18)$$

The last summation is absolutely convergent, and hence holomorphic, in $\Re(s_1 + s_2) > -l$.

Proof of (1) of Theorem 1.4. By putting $l = 0$ in (2.18), we have

$$\begin{aligned} \zeta_2(s_1, s_2; a) &= a^{-s_1} (\zeta(s_2, a) - a^{-s_2}) + \frac{\zeta(s_1 + s_2 - 1, a) - a^{1-s_1-s_2}}{s_2 - 1} \\ &\quad - \frac{\zeta(s_1 + s_2, a) - a^{-s_1-s_2}}{2} - \sum_{n=1}^{\infty} \frac{\Phi_0(s_2 | n, a)}{(n+a)^{s_1}}. \end{aligned}$$

Note that the last sum converges when $\Re(s_1 + s_2) > 0$. Hence we have

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \left(\frac{\zeta(s_1 + s_2, a) - a^{-s_1-s_2}}{2} + \sum_{n=1}^{\infty} \frac{\Phi_0(s_2 | n, a)}{(n+a)^{s_1}} \right) = 0 \quad (2.19)$$

for $\Re(s_1 + s_2) > 1 + \delta$, where $\delta > 0$. By using (2.7) and (2.19), we have

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \zeta_2(1 - 2\varepsilon, 1 + \varepsilon; a) = \lim_{\varepsilon \rightarrow +0} (\varepsilon a^{2\varepsilon-1} \zeta(1 + \varepsilon, a) + (\zeta(1 - \varepsilon, a) - a^{\varepsilon-1})) = -\infty$$

since $1 - 2\varepsilon + 1 + \varepsilon > 1 + \delta$ for some $\delta > 0$. Thus one has

$$\lim_{\varepsilon \rightarrow +0} \zeta_2(1 - 2\varepsilon, 1 + \varepsilon; a) = -\infty.$$

First suppose $0 < a < 1/2$. Then we have $\zeta(0, a) = 1/2 - a > 0$. Hence there exists $0 < \sigma_0 < 1$ such that $\zeta(\sigma_0, a) > 0$ when $0 < a < 1/2$. Then one has

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \zeta_2(\sigma_0 - \varepsilon, 1 + \varepsilon; a) = a^{-\sigma_0} + (\zeta(\sigma_0, a) - a^{-\sigma_0}) = \zeta(\sigma_0, a) > 0$$

from (2.7) and (2.19). Hence $\lim_{\varepsilon \rightarrow +0} \zeta_2(\sigma_0 - \varepsilon, 1 + \varepsilon; a) = +\infty$ when $0 < a < 1/2$. Therefore $\zeta_2(s_1, s_2; a)$ has at least one real zero in $0 < \sigma_1 < 1$, $\sigma_2 > 1$ and $1 < \sigma_1 + \sigma_2 < 2$ when $0 < a < 1/2$ by the intermediate value theorem.

Next suppose $a \geq 1/2$. By Lemma 2.2, we obtain $H(a, x + y) < 0$ and $H(1, y) < 0$ for all $x, y > 0$ when $a \geq 1/2$. Hence one has

$$\zeta_2(\sigma_1, \sigma_2; a) < 0, \quad 0 < \sigma_1 < 1, \quad \sigma_2 > 1, \quad 1 < \sigma_1 + \sigma_2 < 2 \quad (2.20)$$

when $a \geq 1/2$ from the integral representation (2.16). Therefore, the function $\zeta_2(\sigma_1, \sigma_2; a)$ with $a \geq 1/2$ does not vanish when $0 < \sigma_1 < 1$, $\sigma_2 > 1$ and $1 < \sigma_1 + \sigma_2 < 2$. \square

We show the following lemma which is a generalization of Lemma 2.5.

Lemma 2.8. *For $\Re(s_1) > 1$ and $\Re(s_2) > 1$, we have the integral representation*

$$\Gamma(s_1)\Gamma(s_2)\Phi_2(s_1, s_2, a, z_1, z_2) = \int_0^\infty \frac{y^{s_2-1}}{e^y - z_2} \int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - z_1} dx dy. \quad (2.21)$$

Furthermore, we have the following:

- (1). Suppose $z_1 = z_2 = 1$. Then the integral representation (2.21) holds for $\Re(s_1) > 0$, $\Re(s_2) > 1$ and $\Re(s_1 + s_2) > 2$.
- (2). Suppose $z_1 = 1$ and $z_2 \neq 1$. Then the integral representation (2.21) holds for $\Re(s_1) > 1$ and $\Re(s_2) > 0$.
- (3). Suppose $z_1 \neq 1$ and $z_2 = 1$. Then the integral representation (2.21) holds for $\Re(s_1) > 0$ and $\Re(s_2) > 1$.
- (4). Suppose $z_1 \neq 1$ and $z_2 \neq 1$. Then the integral representation (2.21) holds for $\Re(s_1) > 0$ and $\Re(s_2) > 0$.

Proof. The statement (1) has already proved in Lemma 2.5. Thus we only have to show (2), (3) and (4). First assume $\Re(s_1) > 1$ and $\Re(s_2) > 1$. For $y > 0$, we have

$$\int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - z_1} dx = \sum_{m=0}^\infty \int_0^\infty \frac{z_1^m x^{s_1-1}}{e^{(a+m)(x+y)}} dx = \Gamma(s_1) \sum_{m=0}^\infty \frac{z_1^m e^{-(a+m)y}}{(a+m)^{s_1}}.$$

Hence it holds that

$$\int_0^\infty \frac{y^{s_2-1}}{e^y - z_2} \int_0^\infty \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - z_1} dx dy = \Gamma(s_1) \sum_{m=0}^\infty \frac{z_1^m}{(a+m)^{s_1}} \int_0^\infty \frac{y^{s_2-1} e^{-(a+m)y}}{e^y - z_2} dy$$

where the change of the integration and the summation is justified by the method used in the proof of Lemma 2.5. Moreover, one has

$$\int_0^\infty \frac{y^{s_2-1} e^{-(a+m)y}}{e^y - z_2} dy = \sum_{n=1}^\infty \int_0^\infty \frac{z_2^{n-1} y^{s_2-1}}{e^{(a+m+n)y}} dy = \Gamma(s_2) \sum_{n=1}^\infty \frac{z_2^{n-1}}{(a+m+n)^{s_2}}$$

for $\Re(s_2) > 1$. Therefore we obtain (2.21) when $\Re(s_1) > 1$ and $\Re(s_2) > 1$.

Let $z_1 = 1$ and $z_2 \neq 1$. Then the integral representation (2.21) converges absolutely when $\Re(s_1) > 1$ and $\Re(s_2) > 0$ since $e^y - z_2 \neq 0$ for any $y \geq 0$ (see (2.10)). Similarly, the integral (2.21) converges absolutely when $\Re(s_1) > 0$ and $\Re(s_2) > 1$ since $e^{x+y} - z_1 \neq 0$ for all $x, y \geq 0$ when $z_1 \neq 1$ and $z_2 = 1$. Furthermore, the integral (2.21) converges absolutely when $\Re(s_1) > 0$ and $\Re(s_2) > 0$ since $e^y - z_2 \neq 0$ and $e^{x+y} - z_1 \neq 0$ for all $x, y \geq 0$ if $z_1 \neq 1$ and $z_2 \neq 1$. Thus we have this lemma. \square

Proof of (2), (3) and (4) of Theorem 1.4. First, we prove (2) of Theorem 1.4. Namely, suppose $z_1 = 1$ and $z_2 \neq 1$. From (2.21) and Fubini's theorem, we have

$$\Gamma(s_1)\Gamma(s_2)\Phi_2(s_1, s_2, a, 1, z_2) = \int_0^\infty \int_0^\infty \frac{y^{s_2-1}}{e^y - z_2} \frac{x^{s_1-1} e^{(1-a)(x+y)}}{e^{x+y} - 1} dx dy$$

for $\Re(s_1) > 1$ and $\Re(s_2) > 0$ since the integral above converges absolutely. Let $z_2 \in [-1, 1)$. Then we obtain $\Phi_2(\sigma_1, \sigma_2, a, 1, z_2) > 0$ when $\sigma_1 > 1$ and $\sigma_2 > 0$ by the fact that $e^y - z_2 > 0$ and $e^{x+y} - 1 \geq 0$ for any $x, y \geq 0$.

Next assume that z_2 is not real. Then we have $\Im(\Phi_2(\sigma_1, \sigma_2, a, 1, z_2)) \neq 0$ for any $\sigma_1 > 1$ and $\sigma_2 > 0$ by the manner used in the proof of (2.12). Hence we have (2) of Theorem 1.4. The case (3) is proved similarly. Moreover, we can easily show the case (4) when at least one of z_1 and z_2 are real numbers.

Finally, suppose that both z_1 and z_2 are not real. Then we have

$$\Gamma(s_1)\Gamma(s_2)\Phi_2(s_1, s_2, a, z_1, z_2) = \int_0^\infty \int_0^\infty \frac{x^{s_1-1}y^{s_2-1}e^{(1-a)(x+y)}(e^{x+2y} - \overline{z_1}e^y - \overline{z_2}e^{x+y} + \overline{z_1z_2})}{|e^y - z_2|^2|e^{x+y} - z_1|^2} dx dy.$$

Now put $\overline{z_1} := r_1 e^{2\pi i \theta_1}$ and $\overline{z_2} := r_2 e^{2\pi i \theta_2}$, where $0 < r_1, r_2, \theta_1, \theta_2 \leq 1$. Then one has

$$\begin{aligned} & \Re(e^{x+2y} - \overline{z_1}e^y - \overline{z_2}e^{x+y} + \overline{z_1z_2}) \\ &= e^{x+2y} - e^y r_1 \cos(2\pi\theta_1) - e^{x+y} r_2 \cos(2\pi\theta_2) \\ & \quad + r_1 r_2 (\cos(2\pi\theta_1) \cos(2\pi\theta_2) - \sin(2\pi\theta_1) \sin(2\pi\theta_2)), \end{aligned}$$

$$\begin{aligned} & \Im(e^{x+2y} - \overline{z_1}e^y - \overline{z_2}e^{x+y} + \overline{z_1z_2}) \\ &= -e^y r_1 \sin(2\pi\theta_1) - e^{x+y} r_2 \sin(2\pi\theta_2) + r_1 r_2 (\sin(2\pi\theta_1) \cos(2\pi\theta_2) + \sin(2\pi\theta_2) \cos(2\pi\theta_1)) \\ &= -r_1 \sin(2\pi\theta_1) (e^y - r_2 \cos(2\pi\theta_2)) - r_2 \sin(2\pi\theta_2) (e^{x+y} - r_1 \cos(2\pi\theta_1)). \end{aligned}$$

When $\sin(2\pi\theta_1) \sin(2\pi\theta_2) > 0$, we can see that the sign of $\Im(e^{x+2y} - \overline{z_1}e^y - \overline{z_2}e^{x+y} + \overline{z_1z_2})$ does not change even if x and y run through from 0 to ∞ . Therefore it holds that $\Im(\Phi_2(\sigma_1, \sigma_2, a, z_1, z_2)) \neq 0$ for any $\sigma_1 > 0$ and $\sigma_2 > 0$ if $\sin(2\pi\theta_1) \sin(2\pi\theta_2) > 0$. Hence, we only have to show the case $\sin(2\pi\theta_1) \sin(2\pi\theta_2) < 0$. In this case, we have

$$\begin{aligned} & \Re(e^{x+2y} - \overline{z_1}e^y - \overline{z_2}e^{x+y} + \overline{z_1z_2}) \\ & > e^{x+2y} - e^y r_1 \cos(2\pi\theta_1) - e^{x+y} r_2 \cos(2\pi\theta_2) + r_1 r_2 \cos(2\pi\theta_1) \cos(2\pi\theta_2) \\ &= (e^y - r_2 \cos(2\pi\theta_2)) (e^{x+y} - r_1 \cos(2\pi\theta_1)) > 0. \end{aligned}$$

Thus we have $\Re(\Phi_2(\sigma_1, \sigma_2, a, z_1, z_2)) > 0$ for any $\sigma_1 > 0$ and $\sigma_2 > 0$ if $\sin(2\pi\theta_1) \sin(2\pi\theta_2) < 0$. Therefore, we obtain (2), (3) and (4) of Theorem 1.4. \square

Proof of Proposition 1.5. When $\Re(s_1), \Re(s_2) > 1$, we have

$$\begin{aligned} \zeta(s_1, a)\zeta(s_2, a) &= \left(\sum_{m>n\geq 0} + \sum_{n>m\geq 0} + \sum_{m=n\geq 0} \right) \frac{1}{(m+a)^{s_1}(n+a)^{s_2}} \\ &= \zeta_2(s_1, s_2; a) + \zeta_2(s_2, s_1; a) + \zeta(s_1 + s_2, a) \end{aligned}$$

from (2.13) and the view of the harmonic product. Hence one has

$$2\zeta_2(s, s; a) = \zeta(s, a)^2 - \zeta(2s, a)$$

when $\Re(s) > 1$. Note that the equation above gives an analytic continuation of $\zeta_2(s, s; a)$ for $1/2 < \Re(s) < 1$. Then we have

$$\lim_{\sigma \rightarrow 1-0} \zeta_2(\sigma, \sigma; a) = \infty, \quad \lim_{\sigma \rightarrow 1/2+0} \zeta_2(\sigma, \sigma; a) = -\infty.$$

by (2.7). Hence we have Proposition 1.5 by the intermediate value theorem. \square

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